NINE POINT CIRCLE

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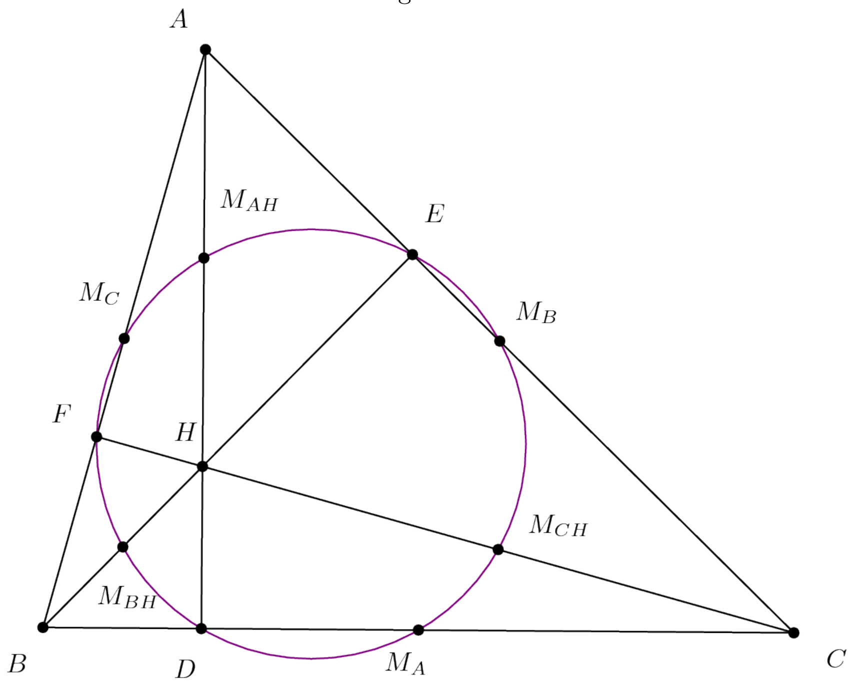
Abstract

I am proud to present one of my first articles concerning Olympiad Geometry. In particular this article is about the Nine Point Circle, some proofs of its existence, properties pertaining to it, and beautiful problems to accompany it all. I hope you will enjoy the article and I wish you a happy reading!

Existence

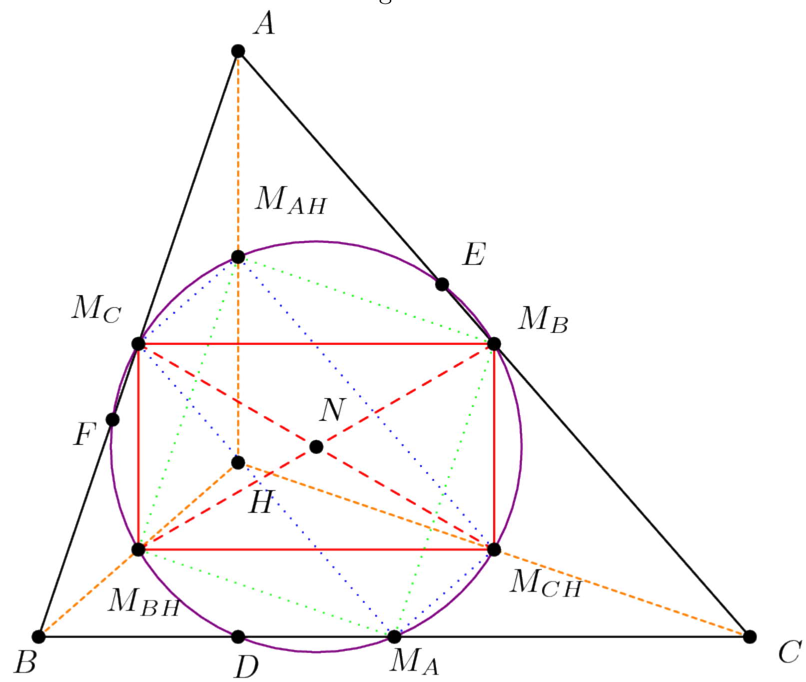
DEFINITION: Consider a triangle ABC with orthocenter H. Let D, E, and F be the feet of the altitudes, let M_A, M_B, and M_C be the midpoints of the sides, and let M_{AH}, M_{BH}, and M_{CH} be the midpoints of AH, BH, and CH respectively, as shown. Then these 9 points are concyclic in the Nine Point Circle, as shown in Figure 1.

Figure 1:



PROOF 1: We shall begin with perhaps the most elementary of proofs. The following proof requires no knowledge of advanced geometry, but rather near-trivial ideas taught in regular high school courses. Consider the following figure:

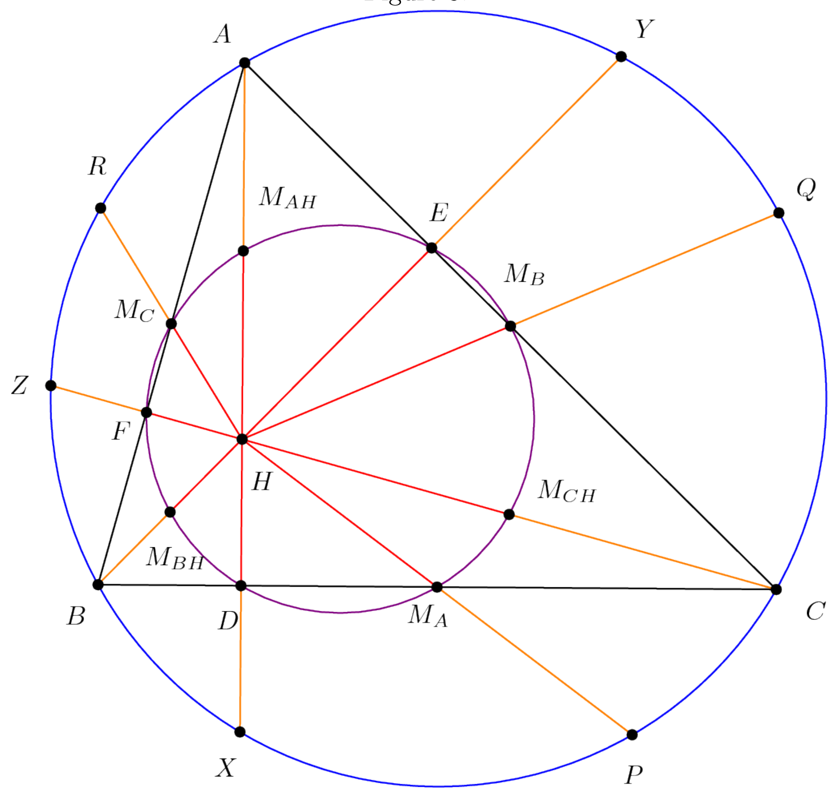
Figure 2:



We know that because M_C and M_{BH} are midpoints of BA and BH, that $M_C M_{BH} \parallel AH$ and $M_C M_{BH} = \frac{1}{2}AH$. Similarly we have $M_B M_{CH} \parallel AH$ and $M_B M_{CH} = \frac{1}{2}AH$. Thus quadrilateral $M_C M_{BH} M_B M_{CH}$ is a parallelogram. However we also know that $M_{BH} M_{CH} \parallel BC$, so because $M_C M_{BH} \parallel AH$ and $AH \perp BC$, we must have $M_C M_{BH} \perp M_{BH} M_{CH}$. Thus quadrilateral $M_C M_{BH} M_B M_{CH}$ is a rectangle. Similarly we find $M_A M_C M_{AH} M_{CH}$ and $M_A M_{BH} M_{AH} M_B$ are rectangles. Thus they are all concylic in a circle with center N where N is the midpoint of $M_X M_{XH}$ for $X \in \{A, B, C\}$. Now we must prove that D, E, and F also lie on this circle. We know that $M_A M_{AH}$ is a diameter of the circle, so any right angle that intercepts that arc must lie on the circle. Hence we have D lies on the circle, and using similar logic we see E and F also lie on the circle. Lastly, we want to prove that N is the midpoint of OH, where O is the circumcenter of $\triangle ABC$. Note that $OM_A = \frac{1}{2}AH = AM_{AH} = M_{AH}H$. Thus $OM_A H M_{AH}$ is a parallegram, and because N is the midpoint of $M_A M_{AH}$, it must also be the midpoint of HO. $\square$

PROOF 2: To prove the existence of the Nine Point Circle, we shall consider the circumcircle of $\triangle ABC$, and a few special points on it.

Figure 3:



We stick to the configuration of Figure 1, but now a few things have been added in (as shown in Figure 3). Namely, these are the circumcircle of ABC (which I will denote with ω), and the points P, Q, R and X, Y, Z. Let P, Q, and R be the intersection of HM_A, HM_B, and HM_C with ω respectively. Let X, Y, and Z be the intersection of HD, HE, and HF with ω respectively. As shown in the diagram above, we will prove that each red segment is the same length as it's corresponding orange segment. For starters, we already defined M_{AH}, M_{BH}, and M_{CH} as the midpoints of AH, BH, and CH respectively, so those 3 segments are taken care of. Now we will use the method of Phantom Pointing. Consider the point T such that T is the reflection of H about BC. Then $\angle BTC = \angle BHC = 180 - A$. So $\angle BTC + \angle BAC = 180$, which implies that T lies on ω, and this further implies that $T \equiv X$. So we know by definition of reflection that $HD = DX$. Similarly we have $HE = EY$ and $HF = FZ$. Now let K be the reflection of H about M_A. Then because $HM_A = M_AK$ and $BM_A = M_AC$, we see that $BHCK$ is a parallelogram, so $\angle BKC = \angle BHC = 180 - A \implies \angle BKC + \angle BAC = 180 \implies K \equiv P$. Thus indeed $HM_A = M_AP$. Similarly we find that $HM_B = M_BQ$ and $HM_C = M_CR$. Now we consider the homothethy with center H and factor $\frac{1}{2}$. This pulls each of the nine points on ω to their respecitive points on the Nine Point Circle. In addition, we see that the Nine Point Center is simply the midpoint of HO.

PROOF 3: There also exists a proof with complex numbers, which although generally I wouldn't present, because it is fairly nice I will give an outline. Consider a triangle abc inscribed in the unit circle. Let $N_9 = \frac{a+b+c}{2}$. Then (using the same notation as in proof 2) we have $H = \frac{a+b+c}{2}$. Also $M_a = \frac{b+c}{2}$. Thus the distance from N_9 to M_A is simply

$$\left|\frac{a+b+c}{2} - \frac{b+c}{2}\right| = \left|\frac{c}{2}\right| = \frac{|c|}{2} = \frac{1}{2}$$

. Similarly we find $N_9M_B = N_9M_C = \frac{1}{2}$. Now we know $M_{AH} = a + \frac{b+c}{2}$, so

$$N_9M_{AH} = \left|\frac{a+b+c}{2} - a - \frac{b+c}{2}\right| = \left|\frac{a}{2}\right| = \frac{1}{2}$$

. Similarly the distance from N_9 to M_{BH} and M_{CH} is $\frac{1}{2}$. Now we use the formula for the foot of an altitude. If K is the foot of the altitude in $\triangle AZB$, where A and B are on the unit circle, then we have

$$k = \frac{1}{2}(a + b + z - ab\bar{z})$$

. Using this we find $D = \dfrac{a+b+c}{2} - \dfrac{ab}{2c}$. So the distance from N_9 to D is

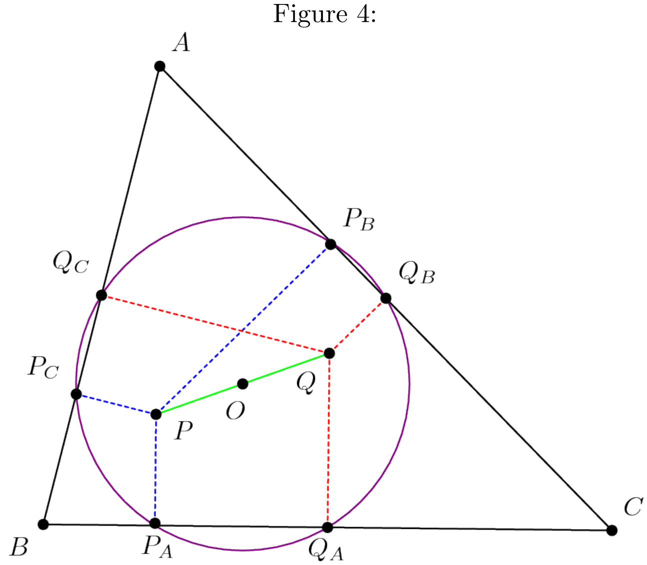
$$\left|\frac{a+b+c}{2} - \frac{a+b+c}{2} + \frac{ab}{2c}\right| = \left|\frac{ab}{2c}\right| = \frac{1}{2}$$

. Similarly we find that the distance from N_9 to E and F is also $\frac{1}{2}$. Thus, all 9 of the aforementioned points are exactly $\frac{1}{2}$ away from N_9, which implies they are concylic in a circle with center N_9. And again, in addition we have that N_9 is the midpoint of HO, because $N_9 = \dfrac{a+b+c}{2}$.

PROOF 4: Finally we arrive at our last proof. It is well known that H and O are isogonal conjugates, so we introduce the following lemma:

lemma

Figure 4:



proof

We want to show (by power of a point) that $AQ_C \cdot AP_C = AP_B \cdot AQ_B$. From there we can apply similar logic to another set of four points to prove the concyclicity of all 6. We note that because P and Q are isogonal conjugates and $\angle PP_CA = \angle QQ_CA = \angle PP_BA = \angle QQ_BA = 90°$, that

$$AQ_C \cdot AP_C = AP \cdot \sin\angle BAP \cdot AQ \cdot \sin\angle BAQ = AP \cdot \sin\angle CAP \cdot AQ \cdot \sin\angle CAQ = AP_B \cdot AQ_B$$

. So the points P_C, Q_C, P_B, and Q_B are concyclic. Repeating this with the other points we find that the six points P_A, P_B, P_C, Q_A, Q_B, and Q_C are concyclic.

Now the center of this circle is the perpendicular bisector of segment P_CQ_C and segment P_BQ_B, which is exactly O.

Getting back to the problem:

Now because H and O are isogonal conjugates we set $P = H$ and $Q = O$. So we see that (using the same notation as proof 1) D, E, F, M_A, M_B, and M_C are concyclic in a circle with center the midpoint of HO. Thus we need only prove that M_{AH}, M_{BH}, and M_{CH} also lie on this circle. To do this we simply use power of a point. We want to show $AM_{AH} \cdot AD = AE \cdot AM_B$. (Note the other points follow similarly, so I will only show that M_{AH} lies on the circle) Or equivalently $AH \cdot AD = AE \cdot AC$. But this is obviously true because quadrilateral $DHEC$ is cyclic. $\square$

Properties

For this section, I will omit the proofs and leave them up to the reader! So go ahead and try to prove these properties as you read through this section, and in case you get stuck make sure to read the hints located towards the end of this article. Good luck!

- The Nine Point Center lies on the Euler Line.
- Given any point on the circumcircle of $\triangle ABC$ the segment formed by connecting that point to the orthocenter of $\triangle ABC$ is bisected by the Nine Point Circle.
- Let N, H, and G be the Nine Point center, Orthocenter, and Centroid of $\triangle ABC$ respectively. Then $HN = 3NG$.
- The Nine Point Circle of $\triangle ABC$ is internally tangent to incircle and externally tangent to the 3 exicrcles at the Feuerbach points of $\triangle ABC$.
- Consider a $\triangle ABC$ and let M_A, M_B, and M_C be the midpoints of sides BC, CA, and AB. The tangents to the Nine Point Circle of $\triangle ABC$ at M_A, M_B, and M_C bound a triangle that is similar to orthic triangle of $\triangle ABC$.
- If $ABCD$ is a cyclic quadrilateral then the Nine Point Circles of triangles ABC, BCD, CDA, and DAB are concurrent at the anticenter of $ABCD$. Furthermore the Nine Point Centers of those triangles form a cyclic quadrilateral that is homothetic to the reference quadrilateral $ABCD$ with ratio $-\frac{1}{2}$.
- Let l be a line passing through the circumcenter O of $\triangle ABC$. Then the orthopole of l lies on the Nine Point Circle of $\triangle ABC$.

Problems

In this section you will encounter around 12 problems I have gathered. Most if not all of these problems are from Art of Problem Solving, written by some wonderful people. I do not take claim for any of these questions. There will be a section at the end with the links to the questions where you will be able to find answers and solutions. And of course, there will be hints provided in the hints section.

1. Let ABC be a triangle and let ω_1 be the circle with diameter BC. Let ω_1 intersect sides CA and AB at points E and F respectively. Let ω_2 be the circumcircle of $\triangle AFE$. A random line through E intersects ω_2 at X and ω_1 at Y. Prove that the midpoint of XY lies on the Nine Point Circle of $\triangle ABC$.

2. Consider a triangle ABC with circumcenter O. Let AD, BE, and CF be altitudes of $\triangle ABC$ and let M_A, M_B, and M_C be the midpoints of sides BC, CA, and AB respectively. Let the tangent to the Nine Point Circle of $\triangle ABC$ at D intersect M_AM_B at D_1. Define E_1 and F_1 similary. Prove that D_1, E_1, and F_1 are collinear on a line perpendicular to the Euler line of $\triangle ABC$.

3. Let ABC be a triangle with orthocenter H. The lines AH, BH, and CH intersect the circumcircle of ABC again at the points D, E, and F respectively. Let A_1 be the reflection of A in the line EF. Define B_1 and C_1 similarly. The line B_1C_1 intersects the side BC again at X; the points Y and Z are similarly defined. Show that X, Y, and Z are collinear on a line which is tangent to the Nine Point Circle of $\triangle DEF$.

4. Let ABC be a triangle with orthocenter H and Nine Point Center N. Let l be the line passing through N perpendicular to AN. Let the feet of the perpendiculars from B, C, and H to l be X, Y, and Z. Prove that $AN + HZ = BX + CY$.

5. Let ABC be a triangle with orthocenter H. Let the centers of $(ABH), (BCH)$, and (CAH) be C_1, A_1, and B_1. Prove that $\triangle A_1B_1C_1 \cong \triangle ABC$ and that their Nine Point Circles coincide.

6. Show that a triangle ABC is right if and only if its Nine Point Circle and circumcircle are tangent.

7. Let ABC be a triangle with circumcenter O. D, E, and F are the centers of $\odot(OBC), \odot(OCA)$, and $\odot(OAB)$, respectively. X, Y, and Z are the reflections of D, E, and F in BC, CA, and AB, respectively. Prove that the Nine Point center of $\triangle XYZ$ lies on the Euler line of $\triangle ABC$.

8. Let $ABCD$ be a cyclic quadrilateral with circumcircle (O). Let AD intersect BC at E, AB intersect CD at F, and AC intersect BD at G.

Prove that the Nine Point Circle of $\triangle GEF$ passes through the centroid of $ABCD$. Furthermore prove that it is the midcircle of (O) and (GEF).

Note: Let (K) and (L) be two circles. Then the midcircle of (K) and (L) is defined to be the circle centered at the midpoint of KL passing through the intersections of (K) and (L).

9. Given two points P and Q on the circumcircle of $\triangle ABC$, prove that the Simson Lines of P and Q intersect on the Nine Point Circle of $\triangle ABC$.

10. Let M_A, M_B, and M_C be the midpoints of sides BC, CA, and AB respectively in $\triangle ABC$. Let H be the orthocenter and let D, E, and F be the midpoints of AH, BH, and CH respectively. Let ω be the Nine Point Circle of $\triangle ABC$. Let the tangents to ω at M_A, M_B, and M_C bound a triangle PQR, and let the tangents to ω at D, E, and F bound a triangle XYZ. Show that the two bounded triangles are congruent.

11. Let I be the incenter of $\triangle ABC$ and let D and E be the intersection of BI and CI with AC and AB respectively. Prove that the Nine Point Center of $\triangle IDE$ lies on AI.

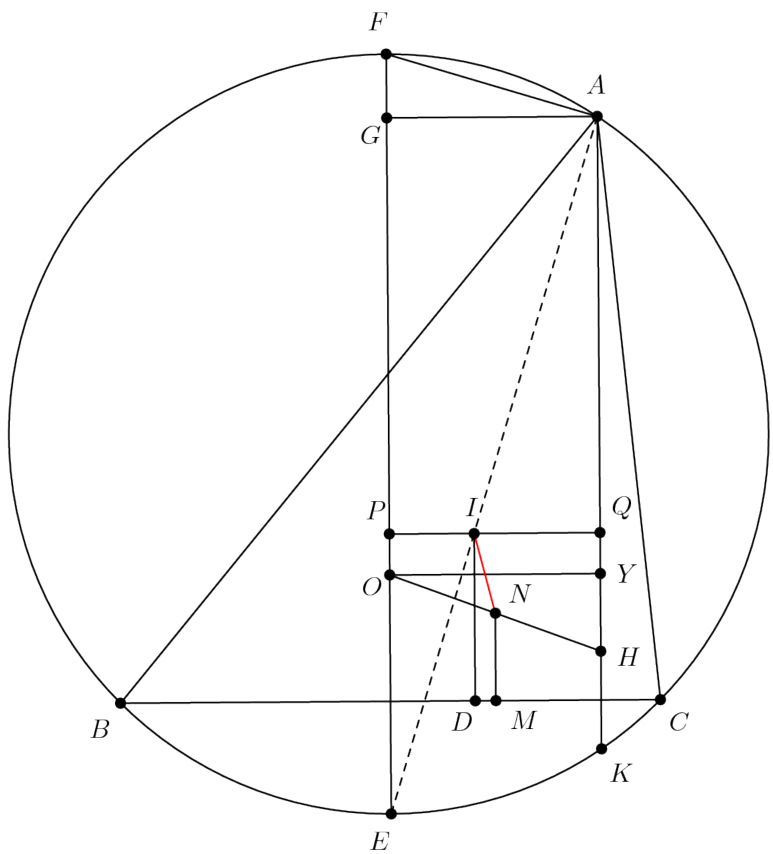
BONUS:

12. Let D, E, and F be the points of contact of the incircle with sides BC, CA, and AB of $\triangle ABC$ respectively. Let D', E', and F' be the reflections of D, E, and F about EF, FD, and DE respectively. Prove that the Nine Point Center of $\triangle ABC$ lies on the Euler line of $\triangle D'E'F'$.

Hints

Properties

1. What is the Nine Point Center the midpoint of?
2. Homothety
3. Recall that $HG = 2GO$.
4. This ones pretty hard, so see if this diagram helps:


F
A
G
P
I
Q
O
Y
N
H
B
D
M
C
K
E

5. Angle Chasing (what do you know about the medial triangle and tangents?)

6. This one's also a little tricky. First thing to realize is that the Nine Point Circles have the same radius. Also, you know that the triangles all have the same circumcenter, so because the Nine Point Center is the midpoint of the segment connecting the orthocenter to the circumcenter, look instead at the orthocenters of the triangles in the problem and try proving those are concyclic.

Problems

1. Try proving that the reflection H' of H about the midpoint of XY lies on the circumcircle. You might want to use Spiral similarity.

2. Radical Axis is a very powerful tool

3. What exactly is the point of tangency? Could it possibly be the Feuerbach Point?

4. Consider the midpoints of AH and BC.

5. What do you know about the radii of $(ABH), (BCH)$, and (CAH)? Try looking for parallelograms, and remember the full definition of Nine Point Circle.

6. Homothety

7. Try finding similar triangles, this should lead you to the conclusion that H is the incenter of $\triangle XYZ$.

8. This question is very hard, I don't think I can give a hint without it either being to trivial or it giving away too much of the problem. Try heading over to the link provided in the next section.

9. What do we know about two Simson Lines of diametrically opposite points? Recall that the Simson Line of a point P bisects the segment PH where H is the orthocenter of $\triangle ABC$.

10. Angle chasing.

11. Introduce some points such as the orthocenter of $\triangle IDE$ into your diagram.

12. The question is bonus for a reason! I myself do not know the proof of this one. This was a question created by the AoPS user TelvCohl. Because it was such a beautiful question I decided to put it on this article. All credits to him of course.

Links to Problems

1. http://artofproblemsolving.com/community/c6h550652
2. http://artofproblemsolving.com/community/c6h536850
3. http://artofproblemsolving.com/community/q2h1069393p4643974
4. http://artofproblemsolving.com/community/q3h550675p3196125
5. http://artofproblemsolving.com/community/c6h364765
6. http://artofproblemsolving.com/community/q3h479763p2686598
7. http://artofproblemsolving.com/community/q3h1090550p4846225
8. http://artofproblemsolving.com/community/q3h595230p3531396
9. http://artofproblemsolving.com/community/c6h488039
10. This question was actually proposed by me, although I don't take credit for it or its originality because it most likely has been though of or posted somewhere before. Anyways, here's the solution:

    Note that the medial triangle is congruent to $\triangle DEF$. Thus the bounded triangles have the same incircle and congruent contact (or intouch) triangles, which means they themselves are congruent. Alternatively we can note that both bounded triangles are similar to the orthic triangle of $\triangle ABC$, and they have the same incircle, so they must be congruent.
11. http://artofproblemsolving.com/community/q5h537441p3088347
12. The link is here: http://artofproblemsolving.com/community/q5h616554p3672663 but as you can see no one has posted a solution yet.